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1996 J. Phys. A: Math. Gen. 29 5873

(http://iopscience.iop.org/0305-4470/29/18/017)

J. Phys. A: Math. Gen. **29** (1996) 5873–5885. Printed in the UK

# Analytic representations based on $SU(1,1)$ coherent states and their applications

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Received 26 March 1996, in final form 6 June 1996

**Abstract.** We consider two analytic representations of the $SU(1,1)$ Lie group: the representation in the unit disc based on the $SU(1,1)$ Perelomov coherent states and the Barut–Girardello representation based on the eigenstates of the $SU(1,1)$ lowering generator. We show that these representations are related through a Laplace transform. A 'weak' resolution of the identity in terms of the Perelomov $SU(1,1)$ coherent states is presented which is valid even when the Bargmann index $k$ is smaller than $\frac{1}{2}$. Various applications of these results in the context of the two-photon realization of $SU(1,1)$ in quantum optics are also discussed.

## 1. Introduction

Analytic representations based on overcomplete sets of quantum states are a mathematical tool used frequently in quantum optics and other areas of quantum physics. These representations have also an important physical meaning, as they provide a natural description of the quantum-classical correspondence. The most familiar example of such a representation is the Bargmann analytic representation (Bargmann 1961) based on the overcomplete set of the Glauber coherent states (Glauber 1963). This representation has also been studied by Fock (1928) and Segal (1962).

We focus here on two well known analytic representations of the $SU(1,1)$ Lie group: one is the analytic representation in the unit disc based on the overcomplete set of the $SU(1,1)$ Perelomov coherent states (Perelomov 1972, 1977, 1986); and the other is the Barut–Girardello representation based on the overcomplete basis of the Barut–Girardello states (Barut and Girardello 1971). These representations are useful in many quantum mechanical problems involving dynamical systems with $SU(1,1)$ symmetry (Kuriyan *et al* 1968, Sharma *et al* 1978, Mukunda *et al* 1980, Sharma *et al* 1981, Kim and Noz 1986). The analytic representation in the unit disc is also related to Berezin's quantization on homogeneous Kählerian manifolds (Berezin 1974, 1975a, b, Perelomov 1986, Bar-Moshe and Marinov 1994). Some other properties and applications of the analytic representation in the unit disc were studied by Gerry (1983), Sudarshan (1993), Vourdas (1992, 1993a, b) and by Brif and Ben-Aryeh (1994b, 1996). The Barut–Girardello representation was used by Basu (1992), Trifonov (1994) and by Prakash and Agarwal (1995).

In the present work we study some interesting properties, relations and applications of these two representations. We show that the Barut–Girardello representation and the analytic representation in the unit disc are related through a Laplace transform. We also consider a resolution of the identity in terms of the $SU(1,1)$ coherent states which is known to be valid only for $k > \frac{1}{2}$. Using an analytic continuation we introduce a 'weak' resolution of the identity which is valid even for $k < \frac{1}{2}$. We apply this method to the two-photon realization of the $SU(1,1)$ Lie algebra in quantum optics, and we obtain the squeezed-state analytic representation. In this context we show that a synthesis of the Barut–Girardello $k = \frac{1}{4}$ and $k = \frac{3}{4}$ representations is related to the Bargmann analytic representation. We also construct analytic representations based on displaced squeezed states and use them to study the energy eigenstates of the squeezed and displaced harmonic oscillator.

## 2. $SU(1,1)$ analytic representations

The group $SU(1,1)$ is the most elementary non-compact non-Abelian simple Lie group. It has several series of unitary irreducible representations: discrete, continuous and supplementary (Bargmann 1947; Vilenkin 1968). The Lie algebra corresponding to the group $SU(1,1)$ is spanned by the three group generators $\{K_1, K_2, K_3\}$:

$$[K_1, K_2] = -\mathrm{i}K_3 \qquad [K_2, K_3] = \mathrm{i}K_1 \qquad [K_3, K_1] = \mathrm{i}K_2. \tag{1}$$

It is customary to use the raising and lowering generators $K_\pm = K_1 \pm \mathrm{i}K_2$, which satisfy

$$[K_3, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_3. \tag{2}$$

The Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ for any irreducible representation is $K^2 = k(k-1)I$. Thus a representation of $SU(1,1)$ is determined by the number $k$. The corresponding Hilbert space $\mathcal{H}_k$ is spanned by the complete orthonormal basis $|n,k\rangle$ $(n = 0, 1, 2, \ldots)$:

$$\langle m,k|n,k\rangle = \delta_{mn} \qquad \sum_{n=0}^{\infty} |n,k\rangle\langle n,k| = I. \tag{3}$$

Various sets of states can be defined in the representation Hilbert space. The overcomplete bases of the $SU(1,1)$ coherent states and of the Barut–Girardello states are of special importance because of their remarkable mathematical properties and interesting physical applications.

### 2.1. Analytic representations in the unit disc

As was discussed by Perelomov (1972, 1977, 1986), each $SU(1,1)$ coherent state corresponds to a point in the coset space $SU(1,1)/\mathrm{U}(1)$ that is the upper sheet of the two-sheet hyperboloid (Lobachevski plane). Thus a coherent state is specified by a pseudo-Euclidean unit vector of the form $(\sinh\tau\cos\varphi, \sinh\tau\sin\varphi, \cosh\tau)$. The coherent states $|\zeta,k\rangle$ are obtained by applying the unitary operators $\Omega(\xi) \in SU(1,1)/\mathrm{U}(1)$ to the lowest state $|n=0,k\rangle$:

$$\begin{aligned} |\zeta,k\rangle &= \exp\left(\xi K_+ - \xi^* K_-\right)|0,k\rangle = (1-|\zeta|^2)^k \exp\left(\zeta K_+\right)|0,k\rangle \\ &= (1-|\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{n!\,\Gamma(2k)}\right]^{1/2} \zeta^n |n,k\rangle. \end{aligned} \tag{4}$$

Here $\xi = -(\tau/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$ and $\zeta = (\xi/|\xi|)\tanh|\xi| = -\tanh(\tau/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$, so $|\zeta| < 1$. The condition $|\zeta| < 1$ shows that the $SU(1,1)$ coherent states are defined in the interior of the unit disc. An important property is the resolution of the identity: for $k > \frac{1}{2}$ one gets

$$\int \mathrm{d}\mu(\zeta,k)\; |\zeta,k\rangle\langle\zeta,k| = I \qquad \mathrm{d}\mu(\zeta,k) = \frac{2k-1}{\pi}\,\frac{\mathrm{d}^2\zeta}{(1-|\zeta|^2)^2} \tag{5}$$

where the integration is over the unit disc $|\zeta| < 1$. For $k = \frac{1}{2}$ the limit $k \to \frac{1}{2}$ must be taken after the integration is carried out in the general form.

One can represent the state space $\mathcal{H}_k$ as the Hilbert space of analytic functions $G(\zeta;k)$ in the unit disc $\mathcal{D}(|\zeta| < 1)$. They form the so-called Hardy space $H_2(\mathcal{D})$. For a normalized state $|\Psi\rangle = \sum_{n=0}^{\infty} C_n|n,k\rangle$, one gets

$$G(\zeta;k) = (1-|\zeta|^2)^{-k}\langle\zeta^*,k|\Psi\rangle = \sum_{n=0}^{\infty} C_n \left[\frac{\Gamma(n+2k)}{n!\Gamma(2k)}\right]^{1/2} \zeta^n \tag{6}$$

$$|\Psi\rangle = \int \mathrm{d}\mu(\zeta,k)\ (1-|\zeta|^2)^k G(\zeta^*;k)|\zeta,k\rangle \tag{7}$$

and the scalar product is

$$\langle\Psi_1|\Psi_2\rangle = \int \mathrm{d}\mu(\zeta,k)\ (1-|\zeta|^2)^{2k}[G_1(\zeta;k)]^* G_2(\zeta;k) \tag{8}$$

This is the analytic representation in the unit disc. The generators $K_\pm$ and $K_3$ act on the Hilbert space of entire functions $G(\zeta;k)$ as first-order differential operators:

$$K_+ = \zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta} + 2k\zeta \qquad K_- = \frac{\mathrm{d}}{\mathrm{d}\zeta} \qquad K_3 = \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + k. \tag{9}$$

The $SU(1,1)$ transformations are easily implemented in this representation through Möbius conformal mappings:

$$\zeta \to \frac{a\zeta + b}{b^*\zeta + a^*} \qquad |a|^2 - |b|^2 = 1. \tag{10}$$

The analytic function is then transformed as

$$G(\zeta;k) \to G\left(\frac{a\zeta+b}{b^*\zeta+a^*};k\right)(b^*\zeta + a^*)^{-2k}. \tag{11}$$

The relation between the parameters $a$, $b$ appearing here and the parameter $\xi = -(\tau/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$ of equation (4) is

$$a = \cosh(\tau/2) \qquad b = \sinh(\tau/2)\,\mathrm{e}^{\mathrm{i}\varphi}. \tag{12}$$

### *2.2. The Barut–Girardello analytic representation*

Barut and Girardello (1971) constructed the eigenstates of the lowering generator $K_-$

$$K_-|z,k\rangle = z|z,k\rangle \tag{13}$$

where $z$ is an arbitrary complex number. The Barut–Girardello states can be decomposed over the orthonormal state basis

$$|z.k\rangle = \frac{z^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n+2k)}}|n,k\rangle \tag{14}$$

where $I_\nu(x)$ is the $\nu$-order modified Bessel function of the first kind. The Barut–Girardello states are normalized but they are not orthogonal to each other. Various properties of these states were studied in different physical contexts, i.e. for different realizations of $SU(1,1)$ (Dodonov *et al* 1974, Sharma *et al* 1978, Mukunda *et al* 1980, Sharma *et al* 1981, Agarwal 1988, Bužek 1990, Basu 1992, Brif and Ben-Aryeh 1994a, Brif 1995). It is not difficult to prove the following identity resolution (Barut and Girardello 1971):

$$\int \mathrm{d}\mu(z,k)\ |z,k\rangle\langle z,k| = I \qquad \mathrm{d}\mu(z,k) = \frac{2}{\pi} K_{2k-1}(2|z|) I_{2k-1}(2|z|)\mathrm{d}^2 z. \tag{15}$$

The integration in (15) is over the whole $z$ plane, and $K_\nu(x)$ is the $\nu$-order modified Bessel function of the second kind. The identity resolution (15) holds for $k > 0$. Thus the Barut–Girardello states form, for each positive $k$, an overcomplete basis in $\mathcal{H}_k$.

The state space $\mathcal{H}_k$ can be represented as the Hilbert space of entire functions $F(z;k)$ which are analytic over the whole $z$ plane. For a normalized state $|\Psi\rangle$, we get

$$F(z;k) = \frac{\sqrt{I_{2k-1}(2|z|)}}{z^{k-1/2}} \langle z^*, k|\Psi\rangle = \sum_{n=0}^{\infty} \frac{C_n}{\sqrt{n!\Gamma(n+2k)}} z^n \tag{16}$$

$$|\Psi\rangle = \int d\mu(z,k)\, \frac{(z^*)^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}} F(z^*;k)|z,k\rangle \tag{17}$$

and the scalar product is

$$\langle\Psi_1|\Psi_2\rangle = \int d\mu(z,k)\, \frac{|z|^{2k-1}}{I_{2k-1}(2|z|)} [F_1(z;k)]^* F_2(z;k). \tag{18}$$

The $SU(1,1)$ generators $K_\pm$ and $K_3$ act on the Hilbert space of entire functions $F(z;k)$ as linear operators:

$$K_+ = z \qquad K_- = 2k\frac{d}{dz} + z\frac{d^2}{dz^2} \qquad K_3 = z\frac{d}{dz} + k. \tag{19}$$

## 3. A Laplace transform between the two analytic representations

It can be easily seen that the unit-disc and Barut–Girardello analytic representations are closely related to each other. We find, for example, that the scalar product between the Perelomov and Barut–Girardello states is

$$\langle\zeta^*, k|z,k\rangle = \left[\frac{z^{k-1/2}(1-|\zeta|^2)^k}{\sqrt{I_{2k-1}(2|z|)\Gamma(2k)}}\right] \exp(z\zeta). \tag{20}$$

For convenience, we introduce the complex number $\rho \equiv 1/\zeta$ ($|\rho| > 1$). Then, according to equation (6), we write

$$G(1/\rho;k) = \sum_{n=0}^{\infty} C_n \left[\frac{\Gamma(n+2k)}{n!\Gamma(2k)}\right]^{1/2} \frac{1}{\rho^n}. \tag{21}$$

Then, by using the expression for the Gamma function

$$\Gamma(n+2k) = \rho^{n+2k} \int_0^\infty t^{n+2k-1} e^{-\rho t} dt \qquad \mathrm{Re}\,\rho > 0 \tag{22}$$

we obtain the following result:

$$G(1/\rho;k) = \frac{\rho^{2k}}{\sqrt{\Gamma(2k)}} \left\{\int_0^\infty \left[z^{2k-1}F(z;k)\right] e^{-\rho z} dz\right\} \qquad \mathrm{Re}\,\rho > 0 \quad |\rho| > 1. \tag{23}$$

This relation gives $G(\zeta;k)$ in the right half of the unit disc ($\mathrm{Re}\,\zeta > 0$), but since $G(\zeta;k)$ is analytic, equation (23) effectively defines $G(\zeta;k)$ in the whole unit disc. Therefore, we see that the two analytic representations are related via the Laplace transform (Erdélyi *et al* 1953b, chapter IV). The inverse Laplace transform (Erdélyi *et al* 1953b, chapter V) yields, correspondingly,

$$F(z;k) = \frac{\sqrt{\Gamma(2k)}}{z^{2k-1}} \left\{\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \left[\rho^{-2k} G(1/\rho;k)\right] e^{\rho z} d\rho\right\}. \tag{24}$$

Note that equations (23) and (24) can also be formally considered as the Mellin transform (Erdélyi *et al* 1953b, chapter VI) and the inverse Mellin transform (Erdélyi *et al* 1953b, chapter VII), respectively. Using the relation (24), we readily find how the $SU(1,1)$ transformations affect the Barut–Girardello representation. Let a normalized state $|\Psi\rangle$ be represented by the functions $G(\zeta;k)$ and $F(z;k)$. If the $SU(1,1)$ transformations are implemented through the Möbius conformal mappings (10), the function $G(\zeta;k)$ is transformed according to (11). Then the relation (24) shows that the function $F(z;k)$ of the form (16) is transformed as

$$F(z;k) \to \left[\frac{\exp(-b^* z/a^*)}{(a^*)^{2k}}\right] \sum_{n=0}^{\infty} \frac{C_n}{\sqrt{n!\Gamma(n+2k)}} R_n(z) \tag{25}$$

where the function $R_n(z)$ can be written by means of the Laguerre polynomial:

$$R_n(z) = (b/a^*)^n n! L_n^{2k-1}(z/a^* b). \tag{26}$$

We see that the function $F(z;k)$ is multiplied by the factor in the square brackets and $z^n$ is replaced by $R_n(z)$.

We now apply the above transform to an example that will be used later. We consider the general eigenvalue equation

$$(\beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3)|\lambda, k\rangle = \lambda|\lambda, k\rangle. \tag{27}$$

Here $\beta_i$ ($i = 1, 2, 3$) are arbitrary complex numbers. In the analytic representation in the unit disc this eigenvalue equation becomes the first-order linear homogeneous differential equation:

$$(\beta_+ + \beta_3\zeta + \beta_-\zeta^2)\frac{\mathrm{d}G(\zeta;\lambda,k)}{\mathrm{d}\zeta} + (2k\beta_-\zeta + k\beta_3 - \lambda)G(\zeta;\lambda,k) = 0 \tag{28}$$

where we have defined $\beta_\pm \equiv \frac{1}{2}(\beta_1 \pm \mathrm{i}\beta_2)$. Equation (28) can be rewritten as

$$(\beta_+\rho^2 + \beta_3\rho + \beta_-)\frac{\mathrm{d}G(1/\rho;\lambda,k)}{\mathrm{d}\rho} - \left(\frac{2k\beta_-}{\rho} + k\beta_3 - \lambda\right)G(1/\rho;\lambda,k) = 0. \tag{29}$$

In the Barut–Girardello analytic representation the eigenvalue equation (27) becomes the second-order linear homogeneous differential equation with linear coefficients:

$$\beta_+ z\frac{\mathrm{d}^2F(z;\lambda,k)}{\mathrm{d}z^2} + (\beta_3 z + 2k\beta_+)\frac{\mathrm{d}F(z;\lambda,k)}{\mathrm{d}z} + (\beta_- z + k\beta_3 - \lambda)F(z;\lambda,k) = 0. \tag{30}$$

It is easy to check that by substituting the Laplace transform (23) into (29), we obtain equation (30); conversely, substituting the inverse Laplace transform (24) into (30), we obtain equation (29). It is interesting to note that the normalization condition for a solution of (30) is equivalent to the analyticity condition for a solution of (29). This condition determines the allowed values of $\lambda$. The technique of analytic representations allows us to obtain the various states associated with the $SU(1,1)$ Lie group and to derive analytic expressions for expectation values over these states.

## 4. A 'weak' resolution of the identity in terms of *SU*(1,1) coherent states

Resolutions of the identity in terms of a certain set of states are very important because they allow the practical use of these states as a basis in the Hilbert space. There are situations where it can be proved that a certain set of states is overcomplete but no resolution of the identity is known. There is a need in these cases to develop weaker concepts than the usual resolutions of the identity, in order to be able to use these states as a basis. The concept of frames used widely in the subject of wavelets is such an example. In the case of the $SU(1,1)$ Perelomov coherent states the resolution of the identity (5) is valid for $k > \frac{1}{2}$ only. In this section we present another resolution of the identity which is valid for both $k > \frac{1}{2}$ and $k < \frac{1}{2}$.

We substitute $\mathrm{d}^2\zeta = \frac{1}{2}\mathrm{d}t\,\mathrm{d}\phi$, (where $\zeta = \sqrt{t}\exp(\mathrm{i}\phi)$) in (5) and integrate over the angle $\phi$, to get

$$I = \sum_{n=0}^{\infty} \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k-1)} \left[\int_0^1 t^n (1-t)^{2k-2}\,\mathrm{d}t\right] |n,k\rangle\langle n,k|. \tag{31}$$

For $k > \frac{1}{2}$, the integral over $t$ is the Euler Beta function. Then one obtains the desired result $I = \sum_{n=0}^{\infty} |n,k\rangle\langle n,k|$. However, for $k < \frac{1}{2}$ the integral in (31) is divergent.

We consider the following relation for the Beta function (Erdélyi *et al* 1953a, section 1.6):

$$\mathrm{B}(x,y) = \frac{1}{2\mathrm{i}\sin(\pi y)} \oint_{\mathcal{C}} t^{x-1}(t-1)^{y-1}\,\mathrm{d}t$$
$$\mathrm{Re}\,x > 0 \quad |\arg(t-1)| \leqslant \pi \quad y \neq 0, \pm 1, \pm 2, \ldots . \tag{32}$$

The contour $\mathcal{C}$ is a single loop that goes from the origin up to one below the real axis, turns back around the point $t = 1$ in the counter-clockwise direction, and goes back above the real axis up to zero.

We next point out that in equation (6) many functions converge in a disc that is larger than the unit disc. We call $H(\mathcal{D}(1+\epsilon))$ the subspace of the Hardy space that contains all the functions that converge in the disc $\mathcal{D}(1+\epsilon) = \{|\zeta| < 1+\epsilon\}$ (where $\epsilon > 0$). Clearly if $\epsilon_1 > \epsilon_2$ then the $H(\mathcal{D}(1+\epsilon_1))$ is a subspace of $H(\mathcal{D}(1+\epsilon_2))$. As $\epsilon$ goes to 0 (from above), the $H(\mathcal{D}(1+\epsilon))$ becomes the Hardy space. Using equation (32), we can prove that for any positive $k$ apart from integers and half-integers and for any two states in $H(\mathcal{D}(1+\epsilon))$ (where $\epsilon$ is any positive number), the scalar product can be written in the form

$$\langle\Psi_1|\Psi_2\rangle = \frac{-(2k-1)\exp(2\pi\mathrm{i}k)}{4\pi\mathrm{i}\sin(2\pi k)} \oint_{\mathcal{C}} \frac{\mathrm{d}t}{(1-t)^{2-2k}} \int_0^{2\pi} \mathrm{d}\phi\,[G_1(\zeta;k)]^* G_2(\zeta;k). \tag{33}$$

The contour $\mathcal{C}$ goes around 1 but is entirely within $\mathcal{D}(1+\epsilon)$. In order to prove this result, we substitute the functions $G(\zeta;k)$ of the form (6) into (33). Taking $\zeta = \sqrt{t}\exp(\mathrm{i}\phi)$, we integrate over the angle $\phi$. Then equation (33) reads

$$\langle\Psi_1|\Psi_2\rangle = \sum_{n=0}^{\infty} C_{1n}^* C_{2n} \frac{\Gamma(n+2k)}{n!\Gamma(2k)} \frac{(2k-1)}{2\mathrm{i}\sin(2\pi k - \pi)} \oint_{\mathcal{C}} t^n (t-1)^{2k-2}\,\mathrm{d}t. \tag{34}$$

We can take the summation out of the integral because the two states belong to $H(\mathcal{D}(1+\epsilon))$, i.e. their analytic functions are convergent on the contour $\mathcal{C}$. We see that the integral in (34) is exactly of the form (32) with $x = n+1$, $y = 2k-1$. Therefore equation (32) can be used for any positive $k$ apart from integers and half-integers. Using the result (32), we rewrite equation (34) in the form

$$\langle\Psi_1|\Psi_2\rangle = \sum_{n=0}^{\infty} C_{1n}^* C_{2n} \frac{\Gamma(n+2k)}{n!\Gamma(2k)} (2k-1)\mathrm{B}(n+1, 2k-1). \tag{35}$$

If we express the Beta function in terms of the Gamma functions, equation (35) reduces to the usual form of the scalar product: $\langle\Psi_1|\Psi_2\rangle = \sum_{n=0}^{\infty} C_{1n}^* C_{2n}$.

Equation (33) is a 'weak' resolution of the identity which we express as

$$I = \frac{-(2k-1)\exp(2\pi\mathrm{i}k)}{4\pi\mathrm{i}\sin(2\pi k)} \oint_{\mathcal{C}} \frac{\mathrm{d}t}{(1-t)^2} \int_0^{2\pi} \mathrm{d}\phi\,|\zeta,k\rangle\langle\zeta,k|. \tag{36}$$

Although the contour $\mathcal{C}$ goes outside the unit disc, where the $SU(1,1)$ coherent states are not normalizable, this equation has to be understood in a weak sense in conjuction with (33). A completeness relation for $SU(1,1)$ coherent states is also discussed by Wünsche (1992) using a rigged Hilbert space.

## 5. Two-photon realization of $SU(1,1)$ in quantum optics

### 5.1. Resolution of the identity

We consider the two-photon realization of the $SU(1,1)$ Lie algebra:

$$K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \qquad K_3 = \tfrac{1}{4}(aa^\dagger + a^\dagger a) \tag{37}$$

where $a$ and $a^\dagger$ are the boson annihilation and creation operators that satisfy the canonical commutation relation $[a, a^\dagger] = I$. The Casimir operator is $K^2 = -\frac{3}{16}$. Therefore, there are two irreducible representations: $k = \frac{1}{4}$ and $k = \frac{3}{4}$ (Bishop and Vourdas 1987). The state space $\mathcal{H}_{1/4}$ is the even Fock subspace with the orthonormal basis consisting of even number eigenstates $|n, \frac{1}{4}\rangle = |2n\rangle$ ($n = 0, 1, 2, \ldots$); the state space $\mathcal{H}_{3/4}$ is the odd Fock subspace with the orthonormal basis consisting of odd number eigenstates $|n, \frac{3}{4}\rangle = |2n+1\rangle$ ($n = 0, 1, 2, \ldots$). A state $|\Psi\rangle$ in the total Fock space can be written as

$$|\Psi\rangle = |\Psi; \mathrm{e}\rangle + |\Psi; \mathrm{o}\rangle = \sum_{n=0}^{\infty} C_n^{(\mathrm{e})}|n; \tfrac{1}{4}\rangle + \sum_{n=0}^{\infty} C_n^{(\mathrm{o})}|n; \tfrac{3}{4}\rangle = \sum_{n=0}^{\infty}(C_{2n}|2n\rangle + C_{2n+1}|2n+1\rangle) \tag{38}$$

with the normalization

$$\sum_{n=0}^{\infty} |C_n^{(\mathrm{e})}|^2 = \mathcal{N}_\mathrm{e} \qquad \sum_{n=0}^{\infty} |C_n^{(\mathrm{o})}|^2 = \mathcal{N}_\mathrm{o} \qquad \mathcal{N}_\mathrm{e} + \mathcal{N}_\mathrm{o} = 1. \tag{39}$$

Here $e$ and $o$ indicate even ($k = \frac{1}{4}$) and odd ($k = \frac{3}{4}$) subspaces, correspondingly. It is clear that $C_{2n} = C_n^{(\mathrm{e})}$, $C_{2n+1} = C_n^{(\mathrm{o})}$.

The unitary group operator $\Omega(\xi) \in SU(1,1)/\mathrm{U}(1)$ for the two-photon realization is the well known squeezing operator $S(\xi)$ (Stoler 1970, 1971, Yuen 1976):

$$S(\xi) = \exp\left(\xi K_+ - \xi^* K_-\right) = \exp\left(\tfrac{1}{2}\xi a^{\dagger 2} - \tfrac{1}{2}\xi^* a^2\right). \tag{40}$$

Therefore, the $SU(1,1)$ coherent states are the squeezed states. For $k = \frac{1}{4}$ one gets the squeezed vacuum

$$|\zeta, \tfrac{1}{4}\rangle = S(\xi)|0\rangle = (1 - |\zeta|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \zeta^n |2n\rangle \tag{41}$$

while for $k = \frac{3}{4}$ one gets the squeezed 'one-photon' state

$$|\zeta, \tfrac{3}{4}\rangle = S(\xi)|1\rangle = (1 - |\zeta|^2)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} \zeta^n |2n+1\rangle. \tag{42}$$

As before, $\zeta = (\xi/|\xi|)\tanh|\xi|$. It is also possible to define the parity-dependent squeezing operator (Brif *et al* 1996) that imposes different squeezing transformations on the even and odd subspaces of the Fock space.

Using the results of the previous section, we obtain the following resolutions of the identity (in a weak sense, as explained above):

$$\frac{1}{8\pi}\oint_{\mathcal{C}}\frac{\mathrm{d}t}{(1-t)^2}\int_0^{2\pi}\mathrm{d}\phi\;|\zeta,\tfrac{1}{4}\rangle\langle\zeta,\tfrac{1}{4}| = \sum_{n=0}^{\infty}|2n\rangle\langle 2n| \tag{43}$$

$$-\frac{1}{8\pi}\oint_{\mathcal{C}}\frac{\mathrm{d}t}{(1-t)^2}\int_0^{2\pi}\mathrm{d}\phi\;|\zeta,\tfrac{3}{4}\rangle\langle\zeta,\tfrac{3}{4}| = \sum_{n=0}^{\infty}|2n+1\rangle\langle 2n+1| \tag{44}$$

where $\zeta = \sqrt{t}\,\mathrm{e}^{\mathrm{i}\phi}$.

### *5.2. A synthesis of the $k=\frac{1}{4}$ and $k=\frac{3}{4}$ Barut–Girardello representations and its relation to the Bargmann representation*

In the two-photon realization the Barut–Girardello eigenvalue equation takes the form

$$a^2|z,k\rangle = 2z|z,k\rangle. \tag{45}$$

Therefore, it is not difficult to see that the Barut–Girardello states $|z,k\rangle$ coincide with the even and odd coherent states

$$|z,\tfrac{1}{4}\rangle = |\alpha\rangle_{\mathrm{e}} = \frac{1}{\sqrt{2\left(1+\mathrm{e}^{-2|\alpha|^2}\right)}}\,(|\alpha\rangle + |-\alpha\rangle) \tag{46}$$

$$|z,\tfrac{3}{4}\rangle = |\alpha\rangle_{\mathrm{o}} = \frac{1}{\sqrt{2\left(1-\mathrm{e}^{-2|\alpha|^2}\right)}}\,(|\alpha\rangle - |-\alpha\rangle) \tag{47}$$

for $k=\frac{1}{4}$ and $k=\frac{3}{4}$, respectively. Here $|\alpha\rangle$ are the familiar Glauber coherent states (Glauber 1963). The even and odd coherent states were introduced by Dodonov *et al* (1974) and they satisfy the eigenvalue equation

$$a^2|\alpha\rangle_{\mathrm{e,o}} = \alpha^2|\alpha\rangle_{\mathrm{e,o}}. \tag{48}$$

The above comments indicate that there must be a relation between a synthesis of the $k=\frac{1}{4}$ and $k=\frac{3}{4}$ Barut–Girardello representations and the Bargmann representation in the $\alpha$ plane (Bargmann 1961), with $\alpha^2 = 2z$.

In order to demonstrate this explicitly we consider the $k=\frac{1}{4}$ and $k=\frac{3}{4}$ Barut–Girardello representations for the even and odd components, correspondingly, of the state $|\Psi\rangle$ of (38) with $z=\alpha^2/2$:

$$F(\tfrac{1}{2}\alpha^2;\tfrac{1}{4}) = \frac{1}{\sqrt{\mathcal{N}_{\mathrm{e}}}}\sum_{n=0}^{\infty}\frac{C_n^{(\mathrm{e})}}{\sqrt{n!\Gamma(n+\frac{1}{2})}}\left(\frac{\alpha^2}{2}\right)^n = \frac{\pi^{-1/4}}{\sqrt{\mathcal{N}_{\mathrm{e}}}}\sum_{n=0}^{\infty}\frac{C_{2n}}{\sqrt{(2n)!}}\,\alpha^{2n} \tag{49}$$

$$F(\tfrac{1}{2}\alpha^2;\tfrac{3}{4}) = \frac{1}{\sqrt{\mathcal{N}_{\mathrm{o}}}}\sum_{n=0}^{\infty}\frac{C_n^{(\mathrm{o})}}{\sqrt{n!\Gamma(n+\frac{3}{2})}}\left(\frac{\alpha^2}{2}\right)^n = \frac{\pi^{-1/4}}{\sqrt{\mathcal{N}_{\mathrm{o}}}}\left(\frac{\alpha}{\sqrt{2}}\right)^{-1}\sum_{n=0}^{\infty}\frac{C_{2n+1}}{\sqrt{(2n+1)!}}\,\alpha^{2n+1}. \tag{50}$$

It is clear that

$$B(\alpha) = \pi^{1/4}\left[\sqrt{\mathcal{N}_{\mathrm{e}}}F(\tfrac{1}{2}\alpha^2;\tfrac{1}{4}) + \sqrt{\mathcal{N}_{\mathrm{o}}}\frac{\alpha}{\sqrt{2}}F(\tfrac{1}{2}\alpha^2;\tfrac{3}{4})\right] \tag{51}$$

is the Bargmann representation (Bargmann 1961) for the state $|\Psi\rangle$. This relation between the Bargmann and Barut–Girardello analytic representations has been discussed by Basu (1992).

This is a particular case of a more general relation between the Bargmann representation for the para-Bose oscillator characterized by the parameter $h_0 = 2k$ and the $SU(1,1)$ Barut–Girardello representations with the indices $k$ and $k+\frac{1}{2}$ (Sharma *et al* 1978, Mukunda *et al* 1980, Sharma *et al* 1981). It is also well known that in the Bargmann representation $a^\dagger = \alpha$, $a = \mathrm{d}/\mathrm{d}\alpha$, and consequently the $SU(1,1)$ generators are the operators of the form

$$K_+ = \frac{\alpha^2}{2} \qquad K_- = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \qquad K_3 = \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha} + \frac{1}{4} \tag{52}$$

which is the same as equation (19) for the Barut–Girardello representation with $k = \frac{1}{4}$ and $z = \alpha^2/2$.

As an application we consider equation (27) which in the Bargmann representation is a second-order differential equation that has two independent solutions: the first corresponds to $k = \frac{1}{4}$, and the second to $k = \frac{3}{4}$. We illustrate this by considering solutions of equation (30). For $\beta_+ \neq 0$, $\beta_3^2 - \beta_1^2 - \beta_2^2 \neq 0$, two independent solutions are

$$F_{\mathrm{e}}(z = \alpha^2/2; \lambda, k) = \exp\left(\frac{\Delta - \beta_3}{\beta_1 + \mathrm{i}\beta_2}\frac{\alpha^2}{2}\right) \Phi\left(k - \frac{\lambda}{\Delta}; 2k; \frac{-\Delta}{\beta_1 + \mathrm{i}\beta_2}\alpha^2\right) \tag{53}$$

$$F_{\mathrm{o}}(z = \alpha^2/2; \lambda, k) = \exp\left(\frac{\Delta - \beta_3}{\beta_1 + \mathrm{i}\beta_2}\frac{\alpha^2}{2}\right) \alpha^{2-4k}\, \Phi\left(\tilde{k} - \frac{\lambda}{\Delta}; 2\tilde{k}; \frac{-\Delta}{\beta_1 + \mathrm{i}\beta_2}\alpha^2\right) \tag{54}$$

where $\Delta \equiv \sqrt{\beta_3^2 - \beta_1^2 - \beta_2^2}$, $\tilde{k} \equiv 1 - k$, and $\Phi(a; b; x)$ is the confluent hypergeometric function (the Kummer function). The two solutions differ by the factor $\alpha^{2-4k}$ and by the replacement $k \to \tilde{k}$. When $k = \frac{1}{4}$, we have $\tilde{k} = \frac{3}{4}$ and $\alpha^{2-4k} = \alpha$. Thus, if the even solution $F_{\mathrm{e}}$ corresponds to $k = \frac{1}{4}$, the other solution is odd and it corresponds to $\tilde{k} = \frac{3}{4}$.

### 5.3. Analytic representations based on displaced squeezed states

In the preceding section we have shown how an analytic representation can be constructed by using the squeezed vacuum states. However, in quantum optics one usually meets squeezing of the Glauber coherent states. The corresponding states are referred to as the displaced squeezed states:

$$|\zeta, \eta\rangle = D(\eta)S(\xi)|0\rangle. \tag{55}$$

Here $S(\xi)$ is the squeezing operator of equation (40) and $D(\eta) = \exp(\eta a^\dagger - \eta^* a)$ is the displacement operator that generates the Glauber coherent states: $|\alpha\rangle = D(\alpha)|0\rangle$. We would like to construct the analytic representation based on the displaced squeezed states. Therefore we use the displaced two-photon realization of the $SU(1,1)$ Lie algebra:

$$K_+(\eta) = D(\eta)K_+D^{-1}(\eta) = \tfrac{1}{2}(a^\dagger - \eta^*)^2 \tag{56}$$

$$K_-(\eta) = D(\eta)K_-D^{-1}(\eta) = \tfrac{1}{2}(a - \eta)^2 \tag{57}$$

$$K_3(\eta) = D(\eta)K_3D^{-1}(\eta) = \tfrac{1}{2}(a^\dagger - \eta^*)(a - \eta) + \tfrac{1}{4}. \tag{58}$$

Since the above operators are produced by a similarity transformation, the commutation relations and the Casimir operator remain unchanged. The displaced $SU(1,1)$ generators can produce both squeezing and displacing transformations. The orthonormal basis corresponding to the displaced two-photon realization of $SU(1,1)$ consists of the displaced Fock states (de Oliveira *et al* 1990, Wünsche 1991)

$$|n, k\rangle_D = D(\eta)|n, k\rangle = \begin{cases} D(\eta)|2n\rangle & k = \frac{1}{4} \\ D(\eta)|2n+1\rangle & k = \frac{3}{4}. \end{cases} \tag{59}$$

The $SU(1,1)$ coherent states are obtained by the action of the unitary operator

$$S_D(\xi) = D(\eta)S(\xi)D^{-1}(\eta) = \exp[\xi K_+(\eta) - \xi^* K_-(\eta)] \tag{60}$$

on the lowest state $|0,k\rangle_D = D(\eta)|0,k\rangle$. As result, we obtain the displaced squeezed states

$$|\zeta,k\rangle_D = D(\eta)S(\xi)|0,k\rangle = D(\eta)|\zeta,k\rangle. \tag{61}$$

These are the $|\zeta,\eta\rangle$ states of (55) when $k=\frac{1}{4}$; for $k=\frac{3}{4}$ the $SU(1,1)$ coherent states are $|1,\zeta,\eta\rangle = D(\eta)S(\xi)|1\rangle$. Apart from the well known resolution of the identity

$$I = \frac{1}{\pi}\int \mathrm{d}^2\eta\ |\zeta,\eta\rangle\langle\zeta,\eta| \tag{62}$$

we now have another resolution of the identity for the squeezed states:

$$I = \frac{1}{8\pi}\oint_{\mathcal{C}} \frac{\mathrm{d}t}{(1-t)^2}\int_0^{2\pi} \mathrm{d}\phi\,(|\zeta,\eta\rangle\langle\zeta,\eta| - |1,\zeta,\eta\rangle\langle 1,\zeta,\eta|)\,. \tag{63}$$

This result follows immediately from relations (43) and (44). Then a state $|\Psi\rangle$ can be represented as

$$|\Psi\rangle = \frac{1}{8\pi}\oint_{\mathcal{C}} \frac{\mathrm{d}t}{(1-t)^2}\int_0^{2\pi} \mathrm{d}\phi\left[(1-t)^{1/4}G_D(\zeta^*;\tfrac{1}{4})|\zeta,\eta\rangle - (1-t)^{3/4}G_D(\zeta^*;\tfrac{3}{4})|1,\zeta,\eta\rangle\right] \tag{64}$$

where

$$G_D(\zeta;k) = (1-|\zeta|^2)^{-k}\,{}_D\langle\zeta^*,k|\Psi\rangle. \tag{65}$$

We can use this 'displaced' version of the $SU(1,1)$ coherent-state analytic representation for analyzing the spectrum of the squeezed and displaced harmonic oscillator (Zhang *et al* 1990)

$$H = \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \tfrac{1}{2}g a^{\dagger 2} + \tfrac{1}{2}g^* a^2 + f a^\dagger + f^* a. \tag{66}$$

Here $\omega$ is a real positive frequency and $g$, $f$ are arbitrary complex parameters. The Hamiltonian (66) is a linear combination of generators of the maximal symmetry group for the quantum harmonic oscillator (Niederer 1973). This group is the semidirect product of the $SU(1,1)$ group and the Weyl-Heisenberg group (Weyl 1950) whose generators are $a$, $a^\dagger$ and $I$. It is well known (Yuen 1976, Beckers and Debergh 1989, Zhang *et al* 1990) that a quantum state evolved by the Hamiltonian of type (66) is displaced and squeezed. The eigenstates and eigenvalues of the Hamiltonian (66) were studied extensively by Lo (1991b), Nagel (1995) and Wünsche (1995). Eigenvalue problems for Hermitian combinations of $SU(1,1)$ generators were first considered by Lindblad and Nagel (1970) and Solomon (1971). We would like to approach this problem by using the squeezed-state analytic representation.

Using the displaced $SU(1,1)$ generators (56)-(58), the Schrödinger equation $H|E\rangle = E|E\rangle$ for the Hamiltonian (66) can be written in the form

$$[2\omega K_3(\eta) + gK_+(\eta) + g^*K_-(\eta)]|E\rangle = (E+\delta)|E\rangle \tag{67}$$

$$\eta = \frac{gf^* - \omega f}{\omega^2 - |g|^2} \qquad \delta = \frac{\omega|f|^2 - \mathrm{Re}\,(gf^{*2})}{\omega^2 - |g|^2}. \tag{68}$$

Using the squeezed-state analytic representation, we write (67) as a first-order linear differential equation of type (28):

$$(g^* + 2\omega\zeta + g\zeta^2)\frac{\mathrm{d}G_D(\zeta;E,k)}{\mathrm{d}\zeta} + (2kg\zeta + 2k\omega - E - \delta)G_D(\zeta;E,k) = 0 \tag{69}$$

where the analytic function $G_D(\zeta; E, k)$ is defined by (65). The solution of equation (69) is

$$G_D(\zeta; E, k) = G_0(\zeta + \chi_-)^{-k+u}(\zeta + \chi_+)^{-k-u} \tag{70}$$

where $G_0$ is a normalization factor and we have defined

$$\chi_\pm \equiv (\omega \pm \Delta)/g \qquad u \equiv (E + \delta)/(2\Delta) \qquad \Delta \equiv \sqrt{\omega^2 - |g|^2}. \tag{71}$$

Here we should distinguish between the two possibilities: $\omega < |g|$ and $\omega > |g|$. When $\omega < |g|$, then $|\chi_+| = |\chi_-| = 1$ and the system has only a continuous spectrum (Lo 1990, 1991b). We consider the case of $\omega > |g|$ in which $\chi_+ = 1/\chi_-^*$ and the system has a discrete spectrum. In this case $|\chi_-| < 1$ and the analyticity condition requires

$$-k + u = l = 0, 1, 2, \ldots \tag{72}$$

that leads to the quantization condition

$$E = E_l(k) = 2\sqrt{\omega^2 - |g|^2}\,(k + l) - \frac{\omega|f|^2 - \mathrm{Re}\,(gf^{*2})}{\omega^2 - |g|^2}. \tag{73}$$

Then the function $G_D(\zeta; E, k)$ takes the form (with $\chi \equiv \chi_- = 1/\chi_+^*$)

$$G_D(\zeta; l, k) = G_0(\zeta + \chi)^l(\zeta + 1/\chi^*)^{-2k-l}. \tag{74}$$

It is clear that for any value of $(\omega - |g|) > 0$ there exists $\epsilon > 0$ such that this function belongs to the subspace $H(\mathcal{D}(1 + \epsilon))$. The analyticity condition (72) cannot be simultaneously satisfied for $k = \frac{1}{4}$ and $k = \frac{3}{4}$. Therefore, if $G_D(\zeta; l, \frac{1}{4})$ is analytic, $G_D(\zeta; l, \frac{3}{4})$ must be zero (the trivial solution), and vice versa. Thus for $\omega > |g|$ the Hamiltonian (66) has two distinct series of eigenstates and eigenvalues: one for $k = \frac{1}{4}$ and the other for $k = \frac{3}{4}$. The eigenstates depend on the Bargmann index $k$ and therefore they belong to only one of the $SU(1, 1)$ irreducible representations.

It can be seen that the function (74) represents the squeezed and displaced Fock states (Kral 1990, Lo 1991a, b):

$$|n, \zeta_0, \eta\rangle = D(\eta)S(\xi_0)|n\rangle. \tag{75}$$

where $\xi_0 = (s/2)\,\mathrm{e}^{\mathrm{i}\theta}$, $\eta = r\,\mathrm{e}^{\mathrm{i}\vartheta}$ are related to $\omega$, $g$, $f$ by

$$\omega/\Delta = \cosh s \qquad g/\Delta = -\sinh s\,\mathrm{e}^{\mathrm{i}\theta} \qquad f/\Delta = \eta\left(\sinh s\,\mathrm{e}^{\mathrm{i}(\theta - 2\vartheta)} - \cosh s\right) \tag{76}$$

and the integer $n$ is given by

$$n = 2l + 2k - \frac{1}{2} = \begin{cases} 2l & k = \frac{1}{4} \\ 2l + 1 & k = \frac{3}{4}\,. \end{cases} \tag{77}$$

Thus the energy eigenstates corresponding to $k = \frac{1}{4}$ ($\frac{3}{4}$) are the squeezed and displaced even (odd) Fock states.

## 6. Conclusions

Analytic representations exploit the powerful theory of analytic functions in a quantum mechanical context. In this paper we have studied various aspects of the analytic representation in the unit disc and the Barut–Girardello representation. We have shown that the two are related through a Laplace transform. We have also considered the resolution of the identity in terms of the $SU(1, 1)$ Perelomov coherent states which is known to be valid only for $k > \frac{1}{2}$. With an analytic continuation we have derived a 'weak' resolution of the identity which is valid even in the region $k < \frac{1}{2}$.

All these ideas have been applied in the context of squeezed states in quantum optics. We have shown how a synthesis of the $\frac{1}{4}$ and $\frac{3}{4}$ representations is related to the Bargmann representation. We have also considered analytic representations based on displaced squeezed states, and used them for the study of the displaced and squeezed harmonic oscillator. The results demonstrate that apart from their theoretical merit, analytic reprentations can also be useful in practical calculations.

## Acknowledgments

CB thanks Professor M S Marinov for valuable discussions and gratefully acknowledges the financial help from the Technion. AV thanks Professor A Wünsche for helpful discussions on $SU(1,1)$ coherent states and gratefully acknowledges support from the British council in the form of a travel grant. AM was supported by the Fund for Promotion of Research at the Technion, by the Technion—VPR Fund, and by GIF—the German–Israeli Foundation for Research and Development.

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